C2 chapter 11 integration (introduction)

Note Title 17/05/2010

11.1 Indefinite & Definite integrals.

differentiate

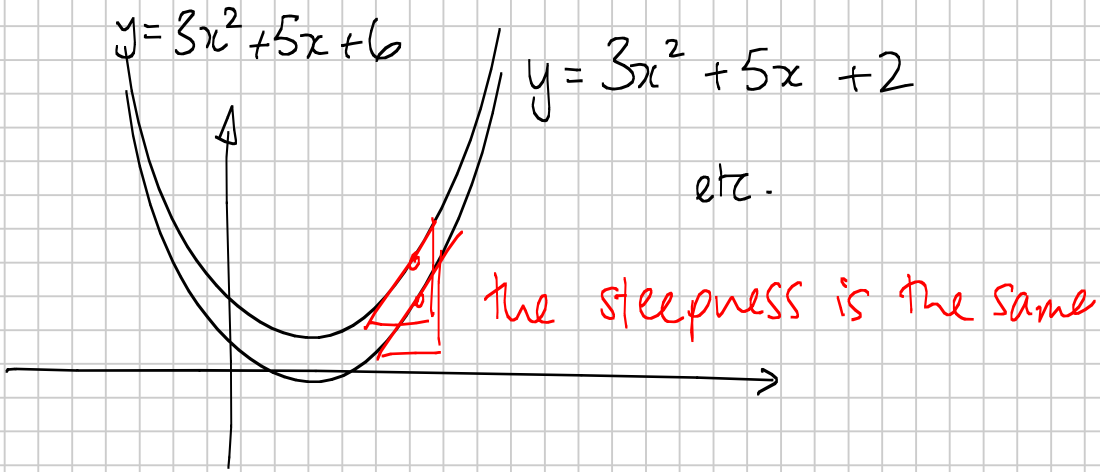
(a) $y = 3x^2 + 5x + 6 \Rightarrow \frac{dy}{dx} = 6x + 5$

(b) $y = 3x^2 + 5x + 2 \Rightarrow \frac{dy}{dx} = 6x + 5$

(c) $y = 3x^2 + 5x - 17325 \Rightarrow \frac{dy}{dx} = 6x + 5$

They're all the same!

What do they look like?



So if I tell you $\frac{dy}{dx} = 6x + 5$ you can tell me

Indefinite integration
anti-differentiation

$$y = \int \frac{dy}{dx}\,dx = \int (6x+5)\,dx = 3x^2 + 5x + C$$

but you don't know what value C was: 6, 2 or -1732.5 or something else.

In C1 we even tried questions where we were told "$\frac{dy}{dx} = 6x + 5$ and (2, 24) lies on curve" we could

find out 'C'

$$y = \int \frac{dy}{dx}\,dx = 3x^2 + 5x + C$$

$$24 = 3(2)^2 + 5(2) + C$$

$$24 = 12 + 10 + C$$

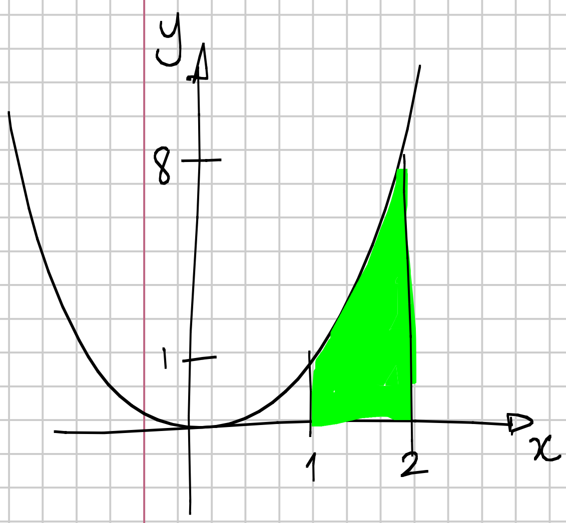
$$C = 2$$

so $y = 3x^2 + 5x + \underline{\underline{2}}$

However, there is another purpose of integration: it also finds the area under a curve.

<u>Definite</u> integration is about finding areas under curves

example



... up to $x=2$

$$\int_1^2 3x^2 \, dx$$

find the area under $y=3x^2$

limits meaning from $x=1$ ...

$$= \left[\frac{3}{3}x^3\right]_1^2$$

notice we don't bother with $+C$

<u>must</u> be square **brackets**

$$= \left[x^3\right]_1^2 = (2)^3 - (1)^3 = 8 - 1 = 7$$

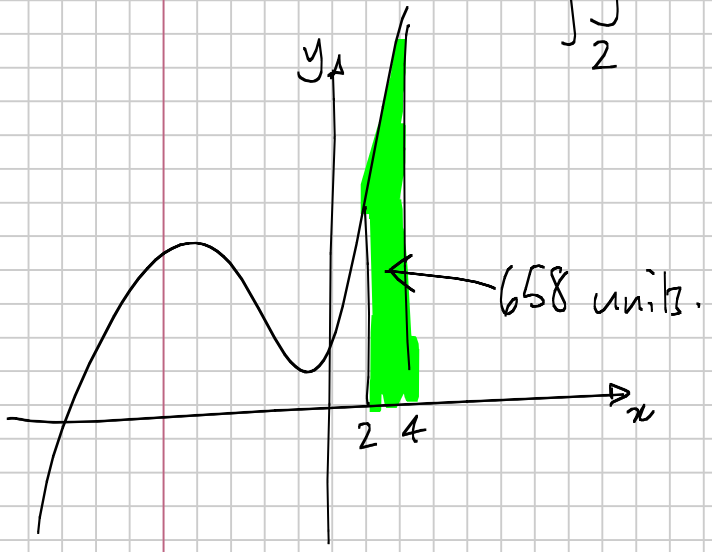
Another example

$$y = 9x^3 + 6x^2 + 3$$

$$\int_2^4 y \, dx = \int_2^4 (9x^3 + 6x^2 + 3) \, dx$$

$$= \left[\frac{9}{4}x^4 + 2x^3 + 3x \right]_2^4$$

not powers just tell you the values of x to use



$$= \left(\frac{9}{4}(4)^4 + 2(4)^3 + 3(4) \right) - \left(\frac{9}{4}(2)^4 + 2(2)^3 + 3(2) \right)$$

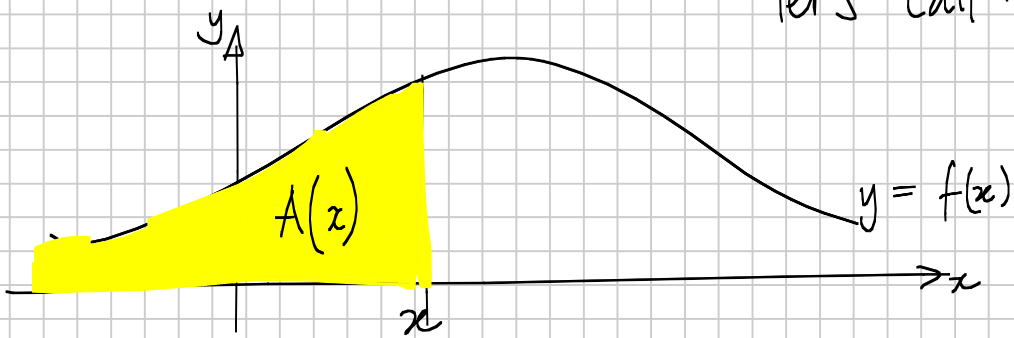
$$= (704 + 12) - (36 + 16 + 6)$$

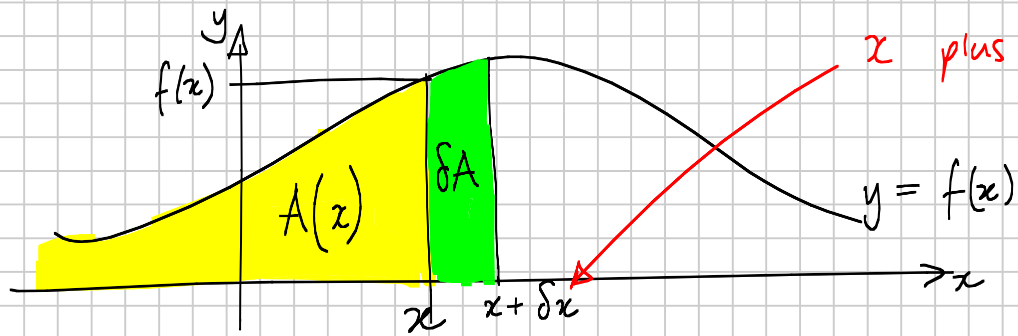
$$= 716 - 58 = 658$$

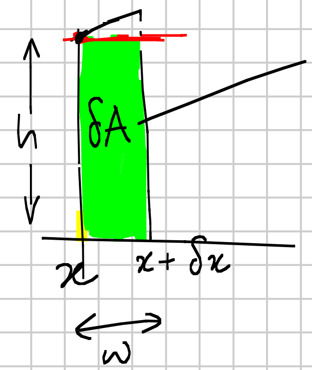
In formal notation $\boxed{\int_a^b f'(x)\,dx = \left[f(x)\right]_a^b = f(b) - f(a)}$

11.2 Why does this work?

let's call the "area so far" $A(x)$







$$\delta A \simeq wh = (\delta x) \times f(x)$$

$$\delta A = y\,\delta x$$

$$\frac{\delta A}{\delta x} = y$$

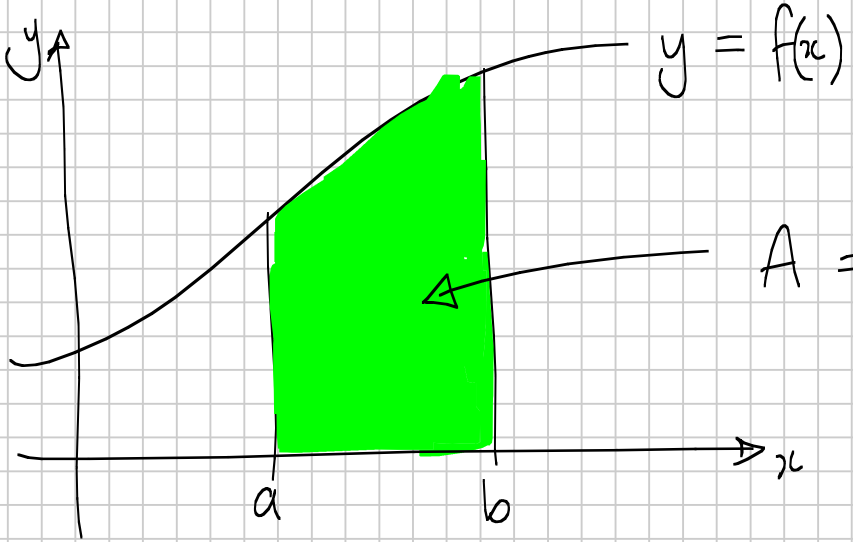
now we take "limits" (which means we think carefully about what will happen as δx gets closer & closer to zero)

$$\lim_{\delta x \to 0} \left(\frac{\delta A}{\delta x} \right) = \frac{dA}{dx} = y$$

now remember $\int$ is an anti-derivative so

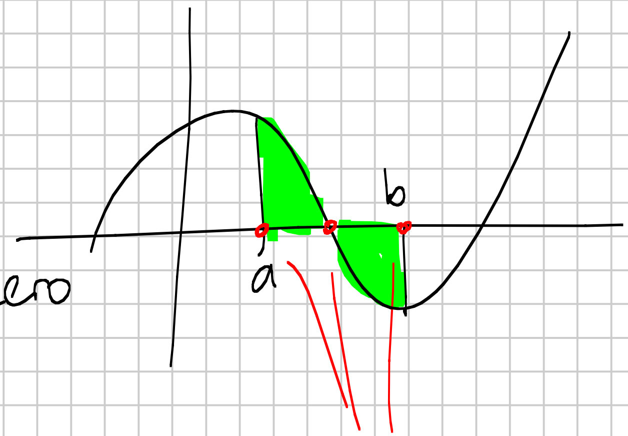
$$\int y \; dx = \int \frac{dA}{dx}\, dx = A$$

The conclusion is



$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx$$

11.3 ⚠ Woah! hold on! What if:

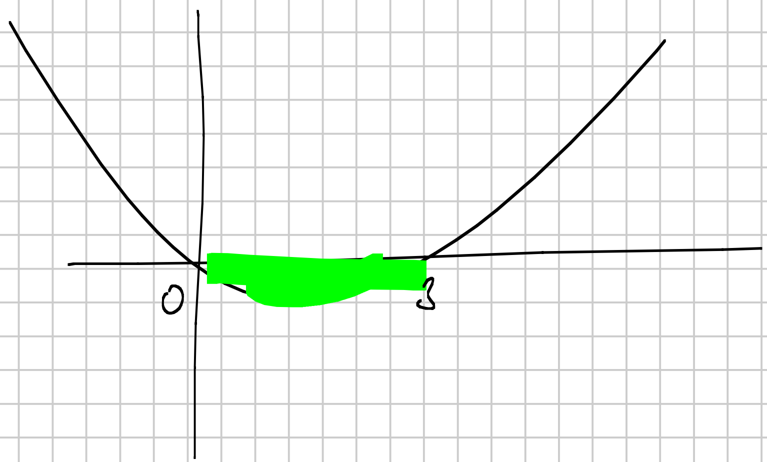


— you might get a negative or zero area and that's silly.

You need to split it and integrate between roots

Example 4 (p162)

$y = x^2 - 3x$ between 0 and 3.



$$\int_0^3 (x^2 - 3x)\,dx$$

$$= \left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_0^3$$

$$= (9 - 13.5) - (0 - 0)$$

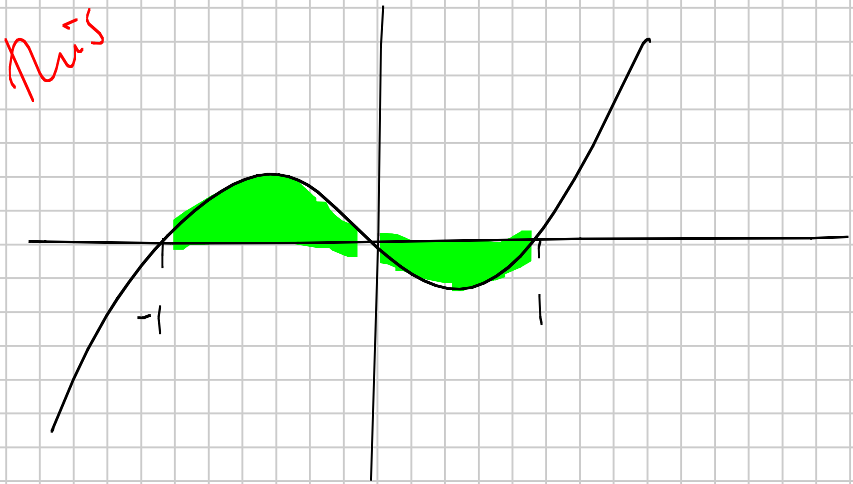
$$= -4.5$$

area $= -4.5$? no! Area $= 4.5$.

⚠ Be careful you integrate without crossing a root:
(Example 46 p162)

Integrate $y = x(x+1)(x-1)$ between -1 and 1.

don't do this



$$\int_{-1}^{1} y \, dx = \int_{-1}^{1} x(x+1)(x-1) \, dx$$

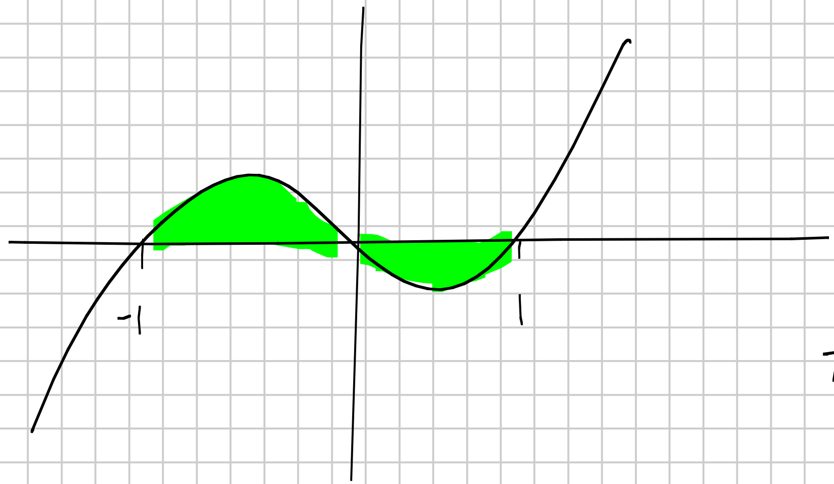
$$= \int_{-1}^{1} (x^3 - x) \, dx$$

$$= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 \right]_{-1}^{1}$$

$$= \left(\frac{1}{4} - \frac{1}{2} \right) - \left(\frac{1}{4} - \frac{1}{2} \right) = 0$$

do this

Integrate $y = x(x+1)(x-1)$ between -1 and 1.



$$A = \int_{-1}^{0} y \, dx - \int_{0}^{1} y \, dx$$

$$A = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{0} - \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{0}^{1}$$

$$A = \left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{4}$$